ANALYSIS OF PARALLEL BEAMS ON ELASTIC FOUNDATION

(RASCHET PARALLEL'NYKH BALOK NA UPRUGOM OSNOVANII)

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A. F. RAKOV
(Berdiansk)

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An approximate solution of a problem of analysis of a system of parallel infinite beams on elastic half-space is given. Prior to this, a spatial contact problem of the theory of elasticity for a half-space is considered, when the contact regions have the forms of parallel strips. The influence of one strip-die on another is investigated in relation to their separation distance, and to their forms. Conditions were found when such influence can be neglected, i.e. when existing theories are applicable for such die systems. Analogous calculations were performed for a system of beams.

The work represents an extension of known results in the area of the theory of beams on elastic foundations obtained primarily by Gorbunov-Posadov [1], Proktor*, Kuznetsov [2] and Rvachev [3-4].

1. It is known that a mixed boundary value problem in the theory of elasticity for a die pressing on an elastic half-space in the absence of friction reduces to the solution of an integral equation

$$w(x, y, 0) = -\frac{1-\nu_0^2}{\pi E_0}\iint_S \frac{p(\xi, \eta)\, d\xi\, d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}} \tag{1.1}$$

where ν_0 and E_0 are elastic constants of the half-space, s is the area of contact; $w(x, y, 0)$ the residue of the boundary points of the half-space in the area of contact; $p(x, y)$ the pressure under the die. If the contact region is determined by an inequality

$$\alpha_i \leqslant x \leqslant \beta_i \quad (i = 1, \ldots, n), \qquad -\infty < y < +\infty \tag{1.2}$$

* Proktor, G.E., On the bending of beams on elastic foundation without the Winkler-Zimmerman hypothesis. Dissertation. A short resume of the dissertation given in [2].

then, it could be shown, for a system of dies whose surfaces after the deformation are represented as

$$z = w_i(x, y, 0) = b_i(\lambda)\cos\lambda y \tag{1.3}$$

where λ is an arbitrary number, and $b_i(\lambda)$ = const, that the pressure can be determined as follows:

$$p(x, y) = \psi_i(\lambda, x)\cos\lambda y \tag{1.4}$$

where $\psi_i(\lambda, x)$ determines the distribution of the pressure under the ith die along its width.

It has been established, using appropriate calculations, that the solution of (1.1) taking into account conditions (1.2), (1.3) and (1.4), reduces to the finding in the plane xoz of some function $\Psi(\lambda, x, z)$ which, everywhere in this plane except perhaps at the segments $\alpha_i \leqslant x \leqslant \beta_i$, $z = 0$, satisfies the equation

$$\nabla_{x,z}^2\Psi - \lambda^2\Psi = 0 \tag{1.5}$$

and the following boundary conditions:

$$\begin{gathered} \Psi(\lambda, x, 0) = b_i(\lambda), \quad \text{if} \quad x \subset [\alpha_i, \beta_i] \\ \left.\frac{\partial\Psi}{\partial z}\right|_{z=0} = 0, \quad \text{if} \quad x \not\subset [\alpha_i, \beta_i] \\ \Psi(\lambda, x, z) \to 0, \quad \text{when} \quad x^2 + z^2 \to \infty \end{gathered} \tag{1.6}$$

After the function Ψ is found, the pressure can be determined from (1.4), where function $\psi_i(\lambda, x)$ satisfies

$$\left.\frac{\partial\Psi}{\partial z}\right|_{z=0} = \frac{2(1-\nu_0^2)}{\pi E_0}\psi_i(\lambda, x), \ x \subset [\alpha_i, \beta_i] \tag{1.7}$$

For $n = 1$, functions Ψ and ψ, which, for convenience, in this case will be denoted by Φ and φ, are known, [3], and have the following form:

$$\Phi(\lambda, x, z) = 2\sum_{i=0}^{\infty}\frac{(-1)^i A_0^{(2i)}\mathrm{Fek}_{2i}(\xi, -q)}{\mathrm{Fek}_{2i}(0, -q)}\mathrm{ce}_{2i}(\eta, -q) \tag{1.8}$$

$$\varphi(\lambda, x) = -\frac{E_0}{(1-\nu_0^2)\sqrt{l^2-x^2}}\sum_{\nu=0}^{\infty}\delta_{2\nu}(l\lambda)\cos 2\nu\cos^{-1}\frac{x}{l} \tag{1.9}$$

$$\delta_{2\nu}(l\lambda) = (-1)^\nu\sum_{i=0}^{\infty}\frac{A_0^{(2i)}A_{2\nu}^{(2i)}\mathrm{Fek}_{2i}(0, -q)}{\mathrm{Fek}_{2i}(0, -q)} \tag{1.10}$$

where $2l$ is the width of a die; $\mathrm{ce}_{2i}(\eta, -q)$ and $\mathrm{Fek}_{2i}(\xi, -q)$ are known tabulated Mathieu functions [5]; $A_{2\nu}^{(2i)}$ are Fourier coefficients of functions $ce_{2i}(x, -q)$, $q = 1/4\ l^2\lambda^2$; the variables ξ and η are connected with x and z by the formulas $x = l \cos\eta \cosh\xi$, $z = l \sin\eta \sinh\xi$; Besides in (1.8) and (1.9), $b(\lambda) = 1$.

The contact region is multipli-connected when $n > 1$ which considerably complicates the finding of Ψ. For the determination of the characteristics, however, related to the stress calculation in the beams, it is sufficient to know the average pressures over the widths of the beams. Consequently, we formulate the following problem.

Consider a region in the xoz plane bounded by a contour $L = L_1 + L_2 + \ldots + L_n$, where $L_i = \beta_i - \alpha_i$ are the segments along the x-axis. Let function $\Psi(\lambda, x, z)$ be given by (1.8), which satisfies (1.5) in xoz, and on one segment L_i satisfies boundary conditions (1.6). It is required to determine the function $\Psi(\lambda, x, z)$ which satisfies (1.5) in the same region xoz, and the boundary conditions (1.6) along a whole contour L.

Using Green's formula for the functions $\Phi(\lambda, x, z)$ and $\Psi(\lambda, x, z)$ we obtain

$$\iint_L (\Phi \nabla_{x,z}^2 \Psi - \Psi \nabla_{x,z}^2 \Phi)\, ds = \int_L \left(\Phi \frac{\partial\Psi}{\partial z} - \Psi \frac{\partial\Phi}{\partial z}\right) dL \qquad (1.11)$$

Because of (1.5) the left-hand side of (1.11) is zero. After some calculations and application of the mean-value theorem for integrals, we obtain

$$\sum_{i=1}^{n} \Phi_i \int_{L_i} \frac{\partial\Psi}{\partial z}\, dx = \sum_{i=1}^{n} \int_{L_i} \Psi \frac{\partial\Phi}{\partial z}\, dx \qquad (1.12)$$

Satisfying in (1.12) conditions (1.6) imposed on the function φ on the segments $L_1, \ldots, L_n$, and taking into account (1.7), we obtain

$$\sum_{i=1}^{n} \Phi_i^{(j)} \int_{L_i} \psi_i(\lambda, x)\, dx = b_j(\lambda) \int_{L_j} \varphi(\lambda, x)\, dx \qquad (j = 1, 2, \ldots, n) \qquad (1.13)$$

where $\Phi_i^{(j)}$ is the value of the function Φ at some interior point of segment L_i, which depends on what segment L_j conditions (1.6) are satisfied.

It can be demonstrated that the function $\Phi(\lambda, x, 0)$ approaches the x-axis asymptotically and smoothly (without corner points). Taking into account this fact and the condition (1.6), we obtain

$$\Phi_i^{(j)} = b(\lambda) = 1 \qquad (i = j), \qquad \Phi(\lambda, \alpha_i, 0) > \Phi_i^{(j)} > \Phi(\lambda, \beta_i, 0) \qquad (i \neq j) \quad (1.14)$$

The integrals on the left- and right-hand of the system of equations (1.13) determine the average pressures over the widths of the beams for $n > 1$ and $n = 1$, respectively. The first of these integrals are unknown, and the second, using (1.9), are found from the formulas

$$\int \varphi(\lambda, x)\, dx = -\frac{\pi E_0}{2(1-\nu_0^2)} \delta_0(l_j\lambda) \qquad (1.15)$$

The formulas for the average pressures for some special cases are as follows:

(1) symmetrical problem for two dies

$$\int_{L_1} \psi(\lambda, x)\, dx = \int_{L_2} \psi(\lambda, x)\, dx = \frac{b_{1,2}(\lambda)}{1+\Phi_2^{(1)}} \int_{L_1} \varphi(\lambda, x)\, dx \qquad (1.16)$$

(2) for a system of identical dies and for like settlements

$$\int_{L_1} \psi(\lambda, x)\, dx = \int_{L_3} \psi(\lambda, x)\, dx = \frac{b_{1,3}(\lambda)}{\Theta} \int_{L_{1,3}} \varphi(\lambda, x)\, dx - \frac{b_2(\lambda)\,\Phi_2^{(1)}}{\Theta} \int_{L_2} \varphi(\lambda, x)\, dx \qquad (1.17)$$

$$\int_{L_2} \psi(\lambda, x)\, dx = \frac{(1+\Phi_3^{(1)})\, b_2(\lambda)}{\Theta} \int_{L_2} \varphi(\lambda, x)\, dx - \frac{2\Phi_1^{(2)} b_{1,2}(\lambda)}{\Theta} \int_{L_{1,3}} \varphi(\lambda, x)\, dx \qquad (1.18)$$

$$\Theta = 1 + \Phi_3^{(1)} - 2\Phi_2^{(1)}\Phi_1^{(2)}$$

(3) for a system of infinitely many, identical, identically loaded, and equally spaced dies (periodic problem):

$$\int_{L_k} \psi(\lambda, x)\, dx = b(\lambda) \int_{L_j} \varphi(\lambda, x)\, dx \Big/ \sum_{i=-\infty}^{\infty} \Phi_i^{(j)} \qquad (1.19)$$

where j is an arbitrary fixed index.

The right-hand sides of (1.16) to (1.19) contain unknown quantities $\Phi_i^{(j)}$ which, however, satisfy (1.14). Using these conditions the corresponding inequality can be obtained for the average pressures.

It can be shown that in the periodical case, as well as in the symmetrical case for two dies, the following inequalities are true:

$$\tfrac{1}{2}[\Phi(\lambda, \alpha_i, 0) + \Phi(\lambda, \beta_i, 0)] \geqslant \Phi_i^{(j)} \geqslant \Phi(\lambda, \tfrac{1}{2}[\alpha_i + \beta_i], 0) \qquad (1.20)$$

The inequalities (1.20) permit us to narrow considerably the approximating range, which contains the values of the average pressures. These inequalities for the case of two dies are

$$\frac{\pi^2 E_0 b(\lambda)\,\delta_0(l\lambda)}{(1+\nu_0^2)\{1+\Phi(\lambda, {}^1\!/_2[\alpha_i+\beta_i], 0)\}} > \tag{1.21}$$

$$> \int\limits_{L_{1,2}} \psi(\lambda, x)\,dx > \frac{\pi^2 E_0 b(\lambda)\,\delta_0(l\lambda)}{(1+\nu_0^2)\{1+{}^1\!/_2[\Phi(\lambda, \alpha_i, 0)+\Phi(\lambda, \beta_i, 0)]\}} \quad (i = 1,2)$$

and for the periodic case

$$\pi^2 E_0 b(\lambda)\,\delta_0(l\lambda)\Big[\sum_{i=-\infty}^{+\infty} \Phi\Big(\lambda, \frac{\alpha_i+\beta_i}{2}, 0\Big)\Big]^{-1} >$$

$$> \int\limits_{L_k} \psi(\lambda, x)\,dx > \pi^2 E_0 b(\lambda)\,\delta_0(l\lambda)\Big(\sum_{i=-\infty}^{+\infty} \frac{\Phi(\lambda, \alpha_i, 0)+\Phi(\lambda, \beta_i, 0)}{2}\Big)^{-1} \tag{1.22}$$

In the Table the limits of the per cent errors are given, which occur when a true value of an average pressure is replaced by one of the extreme values of the inequality as a function of the values $2h = 0,\ l,\ \ldots,\ 5l$, of the separation between the dies and parameter $l\lambda$ which determines the form of the die base. If, for instance, we take $2l = 2$ then the rows of the Table corresponding to the values $l\lambda = 0.5$, will correspond to the change of the die base along its length determined by the expression $w = b \cos 1/2\, y$. Such dies practically do not exert any influence upon each other for $2h \geqslant 4l$ in the case of two dies, and for $2h \geqslant 5l$ in the case of a periodic problem.

TABLE

Case	$l\lambda$	$2h=0$	l	$2l$	$3l$	$4l$	$5l$
(1) (1.21)	0.5	21	4	2	1	0	0
	1.0	32	3	1	0	0	0
(3) (1.22)	0.5	32	7	5	2	1	0
	1.0	57	7	2	0	0	0

The problem of the determination of an average pressure can be solved exactly if $2h = 0$. For, considering n dies touching each other as one die of width $2ln$, we can determine the function $\varphi(\lambda, x)$ from (1.9).

Then, clearly, the average pressure over the width of the ith die is expressed by

$$\int\limits_{L_i} \varphi(\lambda, x)\,dx \tag{1.23}$$

2. The results obtained in Section 1 can be used for the solution of the problem of a system of infinite parallel beams on elastic foundation. The problem is solved with the following assumptions:

(1) the contact region of the beams and elastic half-space is determined from (1.2),

(2) friction is absent between the beams and the foundation,

(3) the deformations of initially plane beams vary only along the length of the beams, and remain unchanged (straight) over the widths of the beams,

(4) the deformation of each beam is determined from the deformation of its center line.

Let $w_i(y)$ be the deformation of a beam, $q_i(y)$ the vertical load, $r_i(y)$ the foundation reaction per unit length of a beam, E_i Young's modulus, J_i the moment of inertia of a cross-section of a beam about its axis parallel to Ox and passing through the centroid of this cross-section. In this case the known equations for the deformation for each of n beams

$$J_i E_i w_i^{(\mathrm{IV})}(y) = q_i(y) - r_i(y) \qquad (i = 1, \ldots, n) \tag{2.1}$$

are satisfied by putting

$$q_i(y) = a_i(\lambda) \cos \lambda y, \qquad w_i(y) = b_i(\lambda) \cos \lambda y, \qquad r_i(y) = c_i((\lambda) \cos \lambda y$$

where $a_i(\lambda)$, $b_i(\lambda)$ and $c_i(\lambda)$ are constants related by

$$J_i E_i b_i(\lambda) \lambda^4 = a_i(\lambda) - c_i(\lambda) \qquad (i = 1, \ldots, n) \tag{2.2}$$

For the solution of the problem, the system (2.2) containing $2n$ unknowns $b_i(\lambda)$ and $c_i(\lambda)$ is not sufficient. The system of equations (1.13) constitutes additional conditions. Indeed, because of the assumptions made, to the settlement of a beam of the form (1.3) there must correspond a pressure in the contact region in the form expressed in (1.4), where $\psi_i(\lambda, x)$ is an unknown function. The reaction per unit length for each beam is

$$r_i(y) = -\int_{L_i} p(x, y)\, dx = -\int_{L_i} \psi_i(\lambda, x)\, dx \cos \lambda y = c_i(\lambda) \cos \lambda y \tag{2.3}$$

Hence

$$c_i(\lambda) = -\int_{L_i} \psi_i(\lambda, x)\, dx$$

Putting the found values of $c_i(\lambda)$ in (2.2) we obtain

$$J_i E_i b_i(\lambda) \lambda^4 = a_i(\lambda) + \int_{r} \psi_i(\lambda, x)\, dx \qquad (i = 1, \ldots, n) \tag{2.4}$$

Eliminating, finally, from (1.13) and (2.4) the integrals (1.23), we obtain a system of n equations for $b_i(\lambda)$

$$\sum_{i=1}^{n} \Phi_i^{(j)} [J_i E_i b_i(\lambda) \lambda^4 - a_i(\lambda)] = b_j(\lambda) \int_{L_j} \varphi(\lambda, x)\, dx \qquad (j = 1, \ldots, n) \tag{2.5}$$

The integrals on the right-hand side in (2.5) are expressed by (1.15). Solving (2.5) we find $b_i(\lambda)$, and thus the deflections of the axes of the beams:

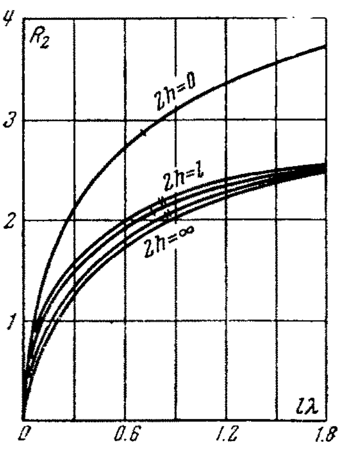
$$w_k(y) = \sum_{i=1}^{n} N_i^{(k)} (\lambda, \Phi_i^{(j)})\, q_i(\lambda) \tag{2.6}$$

where $N_i^{(k)}(\lambda, \Phi_i^{(j)})$ are known functions.

Expressing an arbitrary load by Fourier series, we can write down (2.6) in the following form:

$$w_k(y) = \frac{1}{\pi} \sum_{i=1}^{n} \int_0^{\infty} N_i^{(k)} (\lambda, \Phi_i^{(k)})\, d\lambda \int_{-\infty}^{\infty} q_i(t) \cos \lambda (y - t)\, dt \tag{2.7}$$

It is easy to verify that when the distance between the beams $2h \to \infty$, Formula (2.7) is reduced to a corresponding expression of the deflection for a single beam, found by Rvachev, [3]:



$$w(y) = \frac{1}{\pi JE} \int_0^{\infty} \frac{R_1 d\lambda}{\lambda(\lambda^3 R_1 + lA)} \int_{-\infty}^{\infty} q(t) \cos \lambda (y - t)\, dt \tag{2.8}$$

where

$$R_1 = \frac{l\lambda}{\delta_0(l\lambda)}, \qquad A = \frac{\pi E_0}{JE(1 - \nu_0^2)}$$

For the case of a symmetrical problem for two beams and for infinitely many identical beams, similarly loaded and equally spaced (periodic problem) the necessity to index various quantities vanishes. The deflection formulas in both cases have the same form as (2.8), except for the expression R. For the problem of two beams and periodic problem this function is

$$R_2 = R_1 (1 + \Phi_2^{(1)}), \qquad R_\infty = R_1 \sum_{i=-\infty}^{+\infty} \Phi_i^{(j)} \tag{2.9}$$

where j is an arbitrary fixed index.

On the basis of (1.20) we may obtain for the functions R_2 and R_∞

$$R_1\left[1+\frac{\Phi(\lambda, \alpha_i, 0)+\Phi(\lambda, \beta_i, 0)}{2}\right] > R_2 > R_1\left[1+\Phi\left(\lambda, \frac{\alpha_i+\beta}{2}, 0\right)\right] \qquad (2.10)$$

$$R_1 \sum_{i=-\infty}^{\infty} \frac{\Phi(\lambda, \alpha_i, 0)+\Phi(\lambda, \beta_i, 0)}{2} > R_\infty > R_1 \sum_{i=-\infty}^{\infty} \Phi\left(\lambda, \frac{\alpha_i+\beta_i}{2}, 0\right) \qquad (2.11)$$

These inequalities permit the investigation of a problem of the interaction of beams of a system depending on the value of $2h$. The figure shows the plot of the inequality (2.10). As it could be seen from this figure the curves corresponding to the bounds, and consequently, the curve of the function R_2, are closer to R_1 with increasing distance between the beams. For $2h \geqslant 4l$ the plots of R_2 and R_1 practically coincide. Thus, each beam can be considered independently from another.

In a symmetrical case for three identical and equally loaded beams the deflection curves have the following form:

$$w_{1,2,3}(y) = \frac{1}{\pi IE} \int_0^\infty \frac{R_3 [R_3\lambda^3 + (1+\Phi_2^{(1)} + \Phi_k^{(1)}) A]\, d\lambda}{\Theta(\lambda)} \int_{-\infty}^{+\infty} q(t) \cos\lambda (y-t)\, dt \qquad (2.12)$$

Here $k = 2$ for w_2 and $k = 3$ for w_1 and w_3

$$\Theta(\lambda) = \lambda\, [R_3^2\lambda^6 + (2+\Phi_3^{(1)}) R_3\lambda^3 A + A^2 (1+\Phi_3^{(1)} - 2\Phi_2^{(1)2})] \qquad (2.13)$$

$$R_3 = R_1 (1+\Phi_3^{(1)} - 2\Phi_2^{(1)2})$$

Using (1.14) for R_3 we can obtain a corresponding inequality which permits us to approximate the deflections, and to investigate the interaction of beams.

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